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HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

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VIII.

D. VIEWED IN THE LIGHT OF AN IDEALISTIC CONTINUUM.

The full explanation of Zeno's paradoxes requires two ideas which are very familiar to the modern mathematician, namely, the acceptance of the existence of actually infinite aggregates and the idea of a connected and perfect continuum. The first concept, that of actual infinity, as opposed to potential infinity, had been under contemplation ever since the time of Aristotle. Men like St. Augustine, Galileo, Pascal, Volder, Schultze seemed to have had a more or less non-contradictory conception of it. Others only denied it. Among the latter were both philosophers and mathematicians; the list includes men like Thomas Aquinas, Gerdil, Descartes, Spinoza, Leibniz, Lock, Lotze, Renouvier, Moigno, Cauchy, Gauss, and of course many others.[1] Wonderful insight into this matter was possessed by Galileo. He showed for instance that there were as many integers that were perfect squares as there were integers altogether. Strange to say, Galileo's argument has been misinterpreted by some recent writers. Instead of accepting the conclusion as legitimate for that sort of infinity, as it was accepted by Galileo, these writers declared the conclusion absurd, hence the hypothesis of actual infinity a myth. Among men taking this view was the French philosopher F. Pillon.[2] In 1831 (July 12) the great K. F. Gauss of Göttingen wrote a letter to Schumacher in which he declares himself as opposed to the actual infinity in mathematics.

[1] For details see Georg Cantor, "Ueber die verschiedenen Standpunkte in Bezug auf das actuele Unendliche" in *Zeitschr. f. Philos. u. philos. Kritik*, Bd. 88, p. 224.

[2] *L'année philosophique*, I, 1890, p. 84, quoting among others Cauchy's *Sept leçons de physique générale*, 1868, as follows: "Cette proposition fondamentale, démontrée par Galilée (qu'on ne saurait admettre une suite ou série actuellement composée d'un nombre infini de termes), s'applique aussi bien à une série de termes ou d'objects, qui ont existé." Cauchy was strongly influenced on this matter by the writings of Gerdil.

In the nineteenth century voices in favor of the actual infinity began to speak with greater emphasis. In 1823 John Bolyai, of non-euclidean geometry fame, wrote down that quality of an infinite aggregate: "An infinite aggregate is one equivalent to a part of itself."[1] Another pioneer in this field was the Bohemian mathematician, Bernard Bolzano, whose writings have only in recent years begun to receive proper appreciation.[2] After the appearance of Cantor's writings his ideas received wide recognition among mathematicians, notwithstanding certain perplexing paradoxes to which some of the more advanced developments of the subject gave rise. Among philosophers the Cantorian ideas found slower recognition. The question at issue is usually not so much one of logic, as it is of the postulates which the reasoner is willing to accept as reasonable and useful. An investigator who vetoes any assumption which does not appeal to his intuition or to his power of imagination can hardly find comfort in Cantor's theory of aggregates and the Cantor continuum. To him Zeno's paradoxes must necessarily remain paradoxes forever.

The second notion needed for the full elucidation of our subject is the "connected" and "perfect" continuum, which we owe to Georg Cantor and Dedekind. To their names should be added that of Karl Weierstrass who banished from analysis the mystical notion of the infinitesimal as a constant smaller than any assignable number, defying the Archimedian postulate. We are not aware that any of these three men wrote directly on the paradoxes of Zeno. But they laid the foundation on which a rational theory of them rests. Richard Dedekind brought out two wellknown publications: *Stetigkeit und irrationale Zahlen*, Braunschweig, 1872, and *Was sind und was sollen die Zahlen*, Braunschweig, 1888. Georg Cantor's first important publication on the theory of aggregates is his *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, Leipzig, 1883. The fundamental ideas advanced by Dedekind and Georg Cantor are so easily accessible and so generally known, that no account of them is needed here. At first British mathematicians took little interest in Cantor's developments. Only in recent years have they been taken up in Great Britain. An unusually interesting outline of them is given by Ernest William Hobson in his presidential address "On the Infinite and the Infinitesimal in Mathematical Analysis," before the London Mathematical Society, in 1902.[3] We quote the following:

"When it is conceived that these mere potentialities pass into actualities, that *fixed* numbers or magnitudes exist which are infinite or infinitesimal, that the mere indefinitely great becomes an actual infinite, or the merely indefinitely small becomes an actual infinitesimal, the region of serious controversy has been reached. . . .

"Here we have the origin of the method of limits, in its geometrical and its arithmetical forms, and here we come across the central difficulty of the mode in which a limit was regarded as being actually attained. A limit which appeared only as the unattainable end of a process of indefinite regression, and to which unending approach was made, had, by some process inaccessible to the sensuous imagination, to be regarded as actually reached; the chasm which separated the limit from the approaching magnitudes had in some mysterious way to be leapt over. . . .

[1] See Halsted's *Bolyai's Science Absolute of Space*, 4th Ed., 1896, § 24, p. 20.

[2] See H. Bergmann, *Das philosophische Werk Bernard Bolzanos*, Halle, 1909; also F. Přihonský, *Dr. Bernard Bolzano's Paradoxien des Unendlichen*, Berlin, 1889.

[3] *Proceedings London Mathematical Society*, Vol. 35, London, 1903, p. 117.

"The notion of number, integral or fractional, has been placed upon a basis entirely independent of measurable magnitude, and pure analysis is regarded as a scheme which deals with number only, and has, *per se*, no concern with measurable quantity. Analysis thus placed upon an arithmetical basis is *characterized by the rejection of all appeals to our special intuitions of space, time and motion, in support of the possibility of its operations*. . . ."

"By this conception of the domain of number the root difficulty of the older analysis as to the existence of a limit is turned, each number of the continuum being really defined in such a way that it itself exhibits the limit of certain classes of convergent sequences. . . . It should be observed that the criterion for the convergence of an aggregate is of such a character that no use is made in it of infinitesimals, definite finite numbers alone being used in the test. . . .

"This [old] intuitive notion of the continuum appears to have as its content the notion of unlimited divisibility, the facts that, for instance, in the linear continuum we can within any interval PQ find a smaller one, $P'Q'$, that this process may be continued as far as the limits of our perception allow, and that we are unable to conceive that even beyond the limits of our perception the process of divisibility in thought can come to an end. However, the modern discussions as to the nature of the arithmetic continuum have made it clear that this property of unlimited divisibility, or connexity, is only one of the distinguishing characteristics of the continuum, and is insufficient to mark it off from other domains which have the like property. The aggregate of rational numbers, or of points on a straight line corresponding to such numbers, possess this property of connexity in common with the continuum, and yet it is not continuous." . . .

"The other property of the aggregate which is characteristic of the continuum, is that of being, in the technical language of the theory of aggregates (Mengenlehre) perfect: the meaning of this is that all the limits of the converging sequences of numbers or parts belonging to the aggregate themselves belong to the aggregate; and, conversely, that every number or point of the aggregate can be exhibited as the limit of such a sequence. . . .

". . . the latter property of the continuum, which was not brought to light by those who took the intuitive continuum as a sufficient basis, is in some respects the more absolutely essential property for the domain of a function which is to be submitted to the operation of the calculus."

"In order to exhibit the way in which transfinite ordinal numbers are required when we deal with non-finite aggregates, I propose to refer to a well-known paradox of Achilles and the tortoise. . . . Let us indicate the successive positions of Achilles referred to, by the ordinal numbers 1, 2, 3, . . . suffixed to the letter A, so that $A_1\ A_2\ A_3$ . . . represent the positions of Achilles. . . . These points $A_1\ A_2$

$$\begin{array}{lllllllll} B_2 & & B_3 & B_4 & & & & & \\ \hline A_1 & & A_2 & A_3 & A_4 & A_\omega & A_{\omega+1} & A_{\omega+2} & A_{\omega 2} \end{array}$$

A_3 . . . have a limiting point, which represents the place where Achilles actually catches the tortoise. The limiting point is not contained in the sets of points $A_1\ A_2\ A_3$ . . .; if we wish to represent it, we must introduce a new symbol ω, and denote the point by this number. It does not occur in the series 1, 2, 3, . . . but is preceded by all of these numbers, and yet there is no number immediately preceding it; it is the first of a new series of numbers."

Hobson proceeds to show how a finite number can maintain itself against a transfinite ordinal number, by showing that $\omega = 1 + \omega$, but $\omega + 1 > \omega$; the commutative law in addition is seen to fail. He brings out the necessity for the introduction of transfinite numbers for the representation of the limit which is not itself contained within the region of the convergent process. Hobson's exposition represents an explanation which recent developments of mathematics offer of the "Dichotomy" and the "Achilles." No doubt, some readers might have desired a fuller exposition of details. It will be noticed that the time-element was not considered by Hobson at all. The Dedekind and Georg Cantor theories of the continuum do not involve the element of time. But how is it possible to ignore time in questions involving motion? In the first place it is pointed out by Cantor[1] that the continuum is a much more primitive and general

[1] *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, von Georg Cantor, Leipzig, 1883, p. 29.

concept than the concept of time, that the theory of the continuum is needed for a clear exposition of time or of any independent variable, that time cannot be considered as the measure of motion; on the contrary, time is measured by motion—the motions of heavenly bodies, the motions of the hands of a watch or clock, the displacement of sand in the hour glass. In the second place the consideration of time is not needed at the critical point where the ability of Achilles to overtake the tortoise is under consideration. Suppose the tortoise has an initial start of 10 ft. and that it travels 1 ft. per second, while Achilles travels 10 ft. per second. Forming the series A, whose terms represent each the distance Achilles travels to come up to the place where the tortoise was at the beginning of the time interval under consideration, and letting the series T represent these time-intervals, we have

$$A \qquad 10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots,$$

$$T \qquad 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots.$$

Both geometric infinite series are convergent; the sum of the terms of each series approaches a finite number as a limit. Now comes the ever-present, delicate question whether the sum actually *reaches* its limit. It is to be observed that this question arises in each series, that one series does not help the other. If the sum of A reaches its limit, so does the sum of T, but the possibility of the sum of A reaching its limit is a consideration independent of T. In this sense the consideration of time does not enter the critical part in the explanation of the "Achilles." Whether the sum of A reaches its limit or not is a matter of pure assumption on our part. If the limiting value $11\frac{1}{9}$ ft. is assumed to be included in the aggregate of numbers which the distance-variable may take, then of course the variable reaches its limit; if $11\frac{1}{9}$ ft. is not assumed as a value which the variable may take, then of course the limit is not reached. It is here that we must receive a suggestion from our sensuous observations; we know from our knowledge of motion as supplied to us by our senses that Achilles travelling with a uniform finite velocity in the same direction will within a finite time reach the distance $11\frac{1}{9}$ ft. from his starting point. This information, supplied to us by our senses, enables us to choose, of the two possible alternative assumptions offered by theory (as mentioned above), the one which makes the variable sum A conform with the known sensuous phenomena. On this assumption the region of the convergent process has a limit which is contained in the aggregate of values the variable can take, and, as explained by Hobson, the limit is *reached by the variable.* Viewed from the standpoint of the theory of infinite aggregates and of the Georg Cantor conception of the continuum, the "Achilles" is almost a self-evident proposition. Sensuous knowledge suggests that we make the aggregate of values of the variable distance travelled by Achilles a *perfect* aggregate; then theory tells us that in a perfect aggregate every converging process has a limit which is reached.

Interesting remarks on the Georg Cantor continuum are found also in Hobson's *Theory of Functions of a Real Variable*, Cambridge, 1907, p. 51:

"The term 'arithmetic continuum' is used to denote the aggregate of real numbers, because it is held that the system of numbers of this aggregate is adequate for the complete analytical representation of what is known as continuous magnitude. The theory of the arithmetic continuum has been criticised on the ground that it is an attempt to find the continuous within the domain of number, whereas number is essentially discrete. Such an objection presupposes the existence of some independent conception of the continuum, with which that of the aggregate of real numbers can be compared. At the time when the theory of the arithmetic continuum was developed the only conception of the continuum which was extant was that of the continuum as given by intuition: but this, as we shall show, is too vague a conception to be fitted for an object of exact mathematical thought, until its character as a pure intuitional datum has been modified by exact definitions and axioms."

It will be seen as we proceed that the objection to the Georg Cantor continuum, to which Hobson refers, is frequently made. It is the general objection that a line which is continuous cannot possibly be constructed out of mathematical points, external to each other. Perhaps this general objection which naturally suggests itself at the very start has discouraged non-mathematicians from going to the trouble of studying the Cantor continuum with the care necessary for its comprehension. Philosophers who have subjected themselves to such study have been amply repaid for their labor. They have found it to be a device of the understanding "whereby we give conceptual unity and an invisible connectedness to certain types of phenomenal facts which come to us in a discrete form and in a confused variety."[1]

The other important shift in the point of view, made by the creators of the modern linear continuum, was the rejection of all infinitesimals, that is of quantities which do not obey the Archimedian postulate. This postulate says that if a and b are two numbers (not zero), such that $a < b$, then it is always possible to find a finite integer n so that $na > b$. The infinitesimal which had been the subject of many controversies and was regarded by many as containing an element of mysticism, was banished by Weierstrass and Cantor from their mathematical concepts. In former years the infinitesimal was considered as necessary in the explanation of the linear continuum. Johann Heinrich Lambert wrote to Holland in a letter of April 7, 1766, on the "angle of contact" as follows:[2]

"Do you believe, my dear Sir, that one can dispense with the concept of the infinitely small in the concept of continuity? . . . Continuity demands that this variation be less than every assignable quantity. It is thus impossible to estimate this variation by a finite quantity, and equal to 0 it can not be either. There seems therefore nothing left than to say that the change in direction is infinitely small."

The impossibilities of one generation often become the possibilities of a succeeding generation. Weierstrass's banishment of the infinitely small has found wide following; the old-time infinitesimal is no longer needed in explaining the continuum. The rejection of the infinitely small is looked upon by such mathe-

[1] H. Poincaré, *The Foundations of Science*, transl. by G. B. Halsted, New York, 1913, introduction by Josiah Royce, p. 16.

[2] *J. H. Lamberts deutscher gelehrter Briefwechsel*, Vol. I, Berlin, 1781, p. 141.

matical philosophers and logicians as Bertrand Russell[1] and A. N. Whitehead[2] as steps toward greater mathematical rigor. It must be emphasized, however, that the school of Weierstrass has not found universal recognition; there are modern champions of the infinitely small, chief among whom is the Italian mathematician Giuseppe Veronese. They insist that the Cantor continuum is not the only possible non-contradictory continuum and proceed to construct a higher and more involved, non-archimedean, continuum in which infinitely small distances are given. This is not the place for attempting a minute statement of the controversy between the two schools; the controversy, by the way, has no national aspect. There have been followers of Veronese in Germany (for instance, Stolz, Max Simon), and followers of Weierstrass and G. Cantor in Italy (for instance, Peano). So far as we have noticed, the Zeno arguments have not been studied and given explicit treatment on the basis of the Veronese continuum.[3] In America C. S. Peirce has adhered to the idea of infinitesimals in the declaration: "The illumination of the subject by a strict notation for the logic of relatives had shown me clearly and evidently that the idea of an infinitesimal involves no contradiction."[4] Apparently, before he had acquired familiarity with the writings of Dedekind and Georg Cantor, C. S. Peirce had firmly recognized that for infinite collections the axiom, that the whole is greater than its part, does not hold.

[To be continued]

ON NAPIER'S FUNDAMENTAL THEOREM RELATING TO RIGHT SPHERICAL TRIANGLES.

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In view of the recent celebration of the tercentenary of the publication of Napier's greatest work, the "Mirifici logarithmorum canonis descriptio," it is highly fitting that his rule for the circular parts should be rescued from the rubbish heap of mnemotechnics and be assigned its proper place as the most

[1] See, for instance, his article in the *International Monthly*, Vol. 4, 1901, p. 84 and seq.

[2] A. N. Whitehead, *Introduction to Mathematics*, New York and London, 1911, pp. 156, 226–229.

[3] References to this controversy are as follows: G. Veronese, *Grundzüge der Geometrie von mehreren Dimensionen*, übersetzt v. A. Schepp, Leipzig, 1894, Anhang, p. 631–701; Max Simon, "Historische Bemerkungen über das Continuum, den Punkt und die Gerade Linie," *Atti del IV. Congresso Internazionale dei matematici*, Roma, 1908, pp. 385–390; G. Cantor's letter to Vivanti, *Rivista di mat.* V, 104–108; G. Cantor's letter to Peano, *Rivista di mat.* V, 108–109; G. Cantor, "Zur Begründung der Transfiniten Mengenlehre I," *Mathematische Annalen*, Vol. 46, 1895, page 500; Frederico Enriques, *Probleme der Wissenschaft*, 2. Teil, übersetzt von K. Grelling, Leipzig und Berlin, 1910, pp. 324–329. An able discussion of infinity, infinitesimals and the continuum is given by Josiah Royce, a philosopher familiar with mathematical thought, in his *The World and the Individual*, New York, 1900, pp. 505–560. See also G. Cantor, "Mitteilungen zur Lehre vom Transfiniten" in *Zeitsch. für Philosophie u. Philosophische Kritik*, Vol. 91, Halle, 1887, p. 113; O. Stolz in *Mathematische Annalen*, Bd. XVIII, p. 699, also in *Berichte des naturw.-medizin. Vereins in Innsbruck*, Jahrgänge 1881–82 und 1884, also in *Vorlesungen über allgem. Arithm.*, Leipzig, 1. Theil, 1885, p. 205.

[4] C. S. Peirce, "The Law of Mind" in *The Monist*, Vol. 2, 1892, p. 537.